Wilson–Yukawa Chiral Model on Lattice and Non-commutative Geometry

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Abstract

Based upon the mathematical formulas of Lattice gauge theory and non-commutative geometry differential calculus, we developed an approach of generalized gauge theory on a product of the spacetime lattice and the two discrete points(or a Z_2 discrete group). We introduce a differentiation for non-nearest-neighbour points and find that this differentiation may lead to the introduction of Wilson term in the free fermion Lagrangian on lattice. The Wilson-Yukawa chiral model on lattice is constructed by the generalized gauge theory and a toy model and Smit-Swift model are studied.

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1 Introduction

Although there has been much activity in lattice gauge theory in past years, none of it has been concerned with its geometry framework until Dimakis et.al[1] completed a non-commutative differential calculus on lattice in some sense of non-commutative differential geometry. Their results filled a space of understanding lattice gauge theory in geometry. In recent years, non-commutative geometry has brought a remarkable geometric picture explaining the nature of the Higgs field in the Standard Model[3]—[10]. The electroweak part of the model is explained as originating from the product of the continuous Minkowski space-time by a discrete two-point space. To study the Standard Model further, one must choose an appropriate regularization method. In fact, the lattice provides a very general regularization scheme for quantum field theories and as such has been applied to a variety of models. Several desirable models, one is so called Smit-Swift model[13, 14], were proposed to stimulate the Standard Model in lattice regularization technique. To study these models, fermion fields on lattice must be well defined. It is known that free fermion on lattice meet the "species doubling" problem. One of the most popular schemes dealing with the doubling problem was proposed by Wilson[11]. In this approach, to get rid of the doubling problem, a new term which is called Wilson term was introduced technically in the free fermion lagrangian. This point was accepted extensively so far, but none can explain intrinsically why it exits.

The purpose of this paper is to give a geometrical interpretation for the Wilson term and study how to construct the Wilson–Yukawa Chrial model on lattice from non-commutative differential geometry. In section 2, we develop generalized differential calculus on the product space of spacetime lattice and discrete two–point (or a discrete group Z_2) and introduce a new kind of differentiation for non-nearest-neighbour points. Because non-commutative differential calculus of discrete two–point is equivalent to that of a discrete group Z_2 [2], we will don't distinguish them in the following discussions. In section 3, first we will introduce a free fermion lagrangian on the space of spacetime lattice product a discrete two-point.We find that the non-nearest-neighbour differentiations lead to the introduction of Wilson term. Then we build generalized Yang-Mills-like gauge theory on spacetime lattice product discrete group Z_2 and construct a Wilson-Yukawa chiral model. Smit–Swift model as a special case of the model is discussed in the section 4, then

two examples, a toy model and electroweak model on lattice are studied.

2 Differential calculus on the product of space-time lattice and discrete group Z_2

In this section, we develop the generalized differential calculus on the product of four dimensional lattice and two discrete points(or discrete group Z_2) . For details of differential calculus on lattice and discrete points, it refer to [1, 7].

We introduce a spacetime lattice with lattice spacing a and two discrete points with spacing $\frac{1}{\mu}$. Every point on the lattice is then specified by four integers and an element of discrete group $Z_2 = \{e, Z|Z^2 = e\}$ which we denote collectively by (n, g), $n = (n_1, n_2, n_3, n_4), g \in Z_2$ and n_i are integers.

Let $\mathcal{A}$ be the algebra of complex valued function on spacetime lattice and discrete group Z_2. An element of $\mathcal{A}$ may be written as $f(x, g), f \in \mathcal{A}$, where $X = (x, g)$ is the coordinate , $x_\mu = n_\mu a(\mu = 1, 2, 3, 4)$. Actually, we may also understand discrete group Z_2 as an one dimensional periodic lattice with just two discrete points.

The definition of right action operators on $\mathcal{A}$ is as follows:

$$R_\mu f(x, g) = f(x + a^\mu, g), \mu = 1, 2, 3, 4$$

$$R_Z f(x, g) = f(x, g \cdot Z)$$

where we used the notation

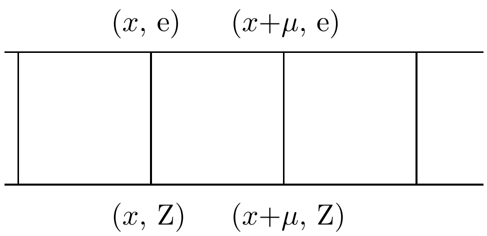
$$(x + a^i)^j = x^j + \delta^j_i a^i. \tag{2.1}$$

It is obvious that

$$R_\mu R_Z f(x, g) = R_Z R_\mu f(x, g) = f(x + a^\mu, g \cdot Z).$$

We introduce discrete partial derivatives on algebra $\mathcal{A}$ as follows:

$$\begin{aligned}
\partial_\mu f(x, g) &= \tfrac{1}{a}(R_i f(x, g) - f(x, g)) \\
\partial_Z f(x, g) &= \mu(R_Z f(x, g) - f(x, g)) \\
\partial_{Z\mu} f(x, g) &= \tfrac{1}{b}(R_\mu R_Z f(x, g) - f(x, g)).
\end{aligned}$$



By the help of above picture, it is easy to show that we defined ∂_μ, ∂_Z by the partial difference between functions of nearest neighbour points and $\partial_{Z\mu}$ by that of non-nearest-neighbour points.In other words, we introduced higher order derivatives on lattice,ie

$$b\partial_{Z\mu} = \frac{a}{\mu}\partial_\mu\partial_Z - a\partial_\mu - \frac{1}{\mu}\partial_Z.$$

Later we will find that the non-nearest-neighbour derivatives are useful to introduce Wilson fermion. In the Euclidean space, there is a relation among spacing parameters as

$$b^2 = a^2 + \frac{1}{\mu^2}.$$

In the following, we recall the notation of differential calculus on algebra $\mathcal{A}$, this is a Z-gradded algebra $\Omega^* = \bigoplus_n \Omega^n$, $\Omega^0 = \mathcal{A}$ the elements of Ω^n are called n-forms.

To complete the construction of the differential calculus, we need to define the exterior derivative d, $d: \ \Omega^n \to \Omega^{n+1}$ whose action on $\mathcal{A}$ is defined by

$$df = \sum\nolimits_\mu \partial_\mu f dx^\mu + \partial_Z f \chi + \sum\nolimits_\mu \partial_{Z\mu} f \chi^\mu, \tag{2.2}$$

where dx^μ, χ, χ^μ are basis of differential one form. It is easy to show that exterior derivative operator d satisfies

$$(i) \qquad d^2 = 0,$$

$$(ii) \qquad d(\omega_1\omega_2) = d\omega_1 \cdot \omega_2 + (-1)^{deg\omega_1}\omega_1 \cdot d\omega_2, \qquad \forall \omega_1, \omega_2 \in \Omega^*,$$

provided that dx^μ, χ and χ^μ satisfy the following conditions

$$\begin{aligned}
dx^\mu f &= R_\mu f dx^\mu \\
\chi f &= R_Z f \chi \\
\chi^\mu f &= R_\mu R_Z f \chi^\mu \\
d(dx^\mu) &= d\chi^\mu = 0 \\
d\chi &= -2\chi\chi.
\end{aligned} \tag{2.3}$$

As a direct results of the above formulas, we have that

$$\begin{aligned}
dx^\mu dx^\nu &= -dx^\nu dx^\mu \\
\chi^\mu \chi^\nu &= -\chi^\nu \chi^\mu \\
dx^\mu \chi &= -\chi dx^\mu \\
dx^\mu \chi^\nu &= -\chi^\nu dx^\mu \\
\chi^\mu \chi &= -\chi \chi^\mu.
\end{aligned}$$

Let us now construct the generalized gauge theory on space-time lattice and discrete group using the above differential calculus. We take the gauge transformations to be any proper subset $\mathcal{H} \subset \mathcal{A}$. In particular, we will often take $\mathcal{H}$ to be unitary elements of $\mathcal{A}$,

$$\mathcal{H} = \mathcal{U}(\mathcal{A}) = \{a \in \mathcal{A} : \quad aa^\dagger = a^\dagger a = 1\}. \tag{2.4}$$

It is easy to see that the exterior derivative d is not covariant with respect to the gauge transformations so that we should introduce the covariant derivative $d + A$, where A is a generalized connection one-form. The requirement that $d+A$ is gauge covariant under gauge transformations

$$d + A \rightarrow H(d + A)H^{-1}, \quad H \in \mathcal{H}, \tag{2.5}$$

results in the following transformation rule of A,

$$A \rightarrow HAH^{-1} + HdH^{-1}. \tag{2.6}$$

If we write $A = \sum_\mu A_\mu dx^\mu + \phi\chi + \sum_\mu B_\mu \chi^\mu$, under gauge transformation $H \in \mathcal{H}$, A_μ, ϕ, B_μ transform as

$$\begin{aligned}
A_\mu &\rightarrow HA_\mu(R_\mu H^{-1}) + H\partial_\mu H^{-1} \\
\phi &\rightarrow H\phi_\mu(R_\mu H^{-1}) + H\partial_Z H^{-1} \\
B_\mu &\rightarrow HB_\mu(R_\mu R_Z H^{-1}) + H\partial_{Z\mu} H^{-1}.
\end{aligned} \tag{2.7}$$

It is convenient to introduce a new field varibles

$$
\begin{aligned}
G_\mu &= 1 + aA_\mu, \mu = 1, 2, 3, 4 \\
\Phi &= \mu + \phi \\
K_\mu &= 1 + bB_\mu, \mu = 1, 2, 3, 4.
\end{aligned}
$$

Then (2.7) is equivalent to

$$
\begin{aligned}
G_\mu &\to HG_\mu(R_\mu H^{-1}) \\
\Phi &\to H\Phi R_Z H^{-1} \\
K_\mu &\to HK_\mu(R_\mu R_Z H^{-1}).
\end{aligned} \tag{2.8}
$$

It can be shown that the generalized curvature two form

$$
F = d\phi + \phi \otimes \phi \tag{2.9}
$$

is gauge covariant and can be written in terms of its coefficients

$$
\begin{aligned}
F = \ & \sum_{\mu,\nu} F_{\mu,\nu} dx^\mu \otimes dx^\nu + \sum_{\mu,\nu} F_{Z\mu,Z\nu} \chi^\mu \otimes \chi^\nu \\
& + \sum_{\mu,\nu} F_{\mu,Z\mu} dx^\mu \otimes \chi^\nu + \sum_{\mu} F_{\mu,Z} dx^\mu \otimes \chi \\
& + \sum_{\mu} F_{Z\mu,Z} \chi^\mu \otimes \chi + F_{Z,Z} \chi \otimes \chi,
\end{aligned} \tag{2.10}
$$

where

$$
\begin{aligned}
F_{\mu,\nu} &= \tfrac{1}{2}\tfrac{1}{a^2}(G_\mu R_\mu G_\nu - G_\nu R_\nu G_\mu) \\
F_{Z\mu,Z\nu} &= \tfrac{1}{2}\tfrac{1}{b^2}(K_\mu R_\mu R_Z K_\nu - K_\nu R_\nu R_Z K_\mu) \\
F_{\mu,Z\mu} &= \tfrac{1}{ab}(G_\mu R_\mu K_\nu - K_\nu R_\mu R_Z G_\mu) \\
F_{\mu,Z} &= \tfrac{1}{a}(G_\mu R_\mu \Phi - \Phi R_Z G_\mu) \\
F_{Z\mu,Z} &= \tfrac{1}{b}(K_\mu R_\mu \Phi - \Phi R_Z K_\mu) \\
F_{Z,Z} &= (\Phi R_Z \Phi - \mu^2).
\end{aligned} \tag{2.11}
$$

In order to get Lagrangian of the Yang-Mills type, we need to define a metrices as follows,

$$
\begin{aligned}
& < dx^\mu, dx^\nu >= g^{\mu\nu}, \qquad < \chi, \chi >= \eta, \qquad < \chi^\mu, \chi^\nu >= \xi g^{\mu,\nu} \\
& < dx^\mu, \chi^\nu >=< \chi^\mu, dx^\nu >= 0, \\
& < dx^\mu, \chi >=< \chi, dx^\nu >= 0, \quad < \chi^\mu, \chi >=< \chi, dx^\nu >= 0 \\
& < dx^\mu \wedge dx^\nu, dx^\sigma \wedge dx^\rho >= \tfrac{1}{2}(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}), \\
& < \chi^\mu \wedge \chi^\nu, \chi^\sigma \wedge \chi^\rho >= \xi^2 \tfrac{1}{2}(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}), \\
& < dx^\mu \otimes \chi^p, dx^\nu \otimes \chi^q >= \xi g^{\mu\nu} g^{pq}, \quad < dx^\mu \otimes \chi, dx^\nu \otimes \chi >= g^{\mu\nu} \eta \\
& < \chi^\mu \otimes \chi, dx^\nu \otimes \chi >= \xi g^{\mu\nu} \eta, \quad < \chi \otimes \chi, \chi \otimes \chi >= \eta^2,
\end{aligned}
\tag{2.12}
$$

Thus, we may introduce the Lagrangian of the Yang-Mills type for the gauge boson and Higgs from the inner product of curvature $< F, \overline{F} >$, where

$$
\begin{aligned}
\overline{F} = \; & \sum_{\mu,\nu} dx^\mu \otimes dx^\nu F^\dagger_{\mu,\nu} + \sum_{\mu,\nu} \chi^\mu \otimes \chi^\nu F^\dagger_{Z\mu,Z\nu} \\
& + \sum_{\mu,\nu} dx^\mu \otimes \chi^\nu F^\dagger_{\mu,Z\mu} + \sum_{\mu} dx^\mu \otimes \chi F^\dagger_{\mu,Z} \\
& + \sum_{\mu} \chi^\mu \otimes \chi F^\dagger_{Z\mu,Z} + \chi \otimes \chi F^\dagger_{Z,Z}.
\end{aligned}
$$

After cumbersome calculation, we get the Lagrangian as follows:

$$
\begin{aligned}
& \mathcal{L}_{YM-H} = -\tfrac{1}{N} < F, \overline{F} > \\
= \; & \tfrac{1}{a^4} \sum_{\mu<\nu} [1 - \frac{1}{2}(U_{\mu,\nu} + U^\dagger_{\mu,\nu})] \\
& + \tfrac{\xi^2}{b^4} \sum_{\mu\nu} \frac{1}{2}[K^\dagger_\mu K_\mu R_\mu R_Z (K_\nu K^\dagger_\nu) - \frac{1}{2}([K^\dagger_\mu K_\nu R_Z (R_\nu K_\mu R_\mu K^\dagger_\nu) + K\nu^\dagger K_\mu R_Z (R_\mu K_\nu R_\nu K^\dagger_\mu)] \\
& \tfrac{\xi}{a^2 b^2} \sum_{\mu\nu} [2 K_\mu K^\dagger_\mu - (G_\mu R_\mu K_\nu R_\nu R_Z G^\dagger_\mu K^\dagger_\nu + K_\nu R_\nu R_Z G_\mu R_\mu K^\dagger_\nu G^\dagger_\mu)] \\
& + \tfrac{\eta}{a^2} \sum_{\mu} [2 \Phi \Phi^\dagger - (G_\mu R_\mu \Phi R_Z G^\dagger_\mu \Phi^\dagger + \Phi R_Z G_\mu R_\mu \Phi^\dagger G^\dagger_\mu)] \\
& + \tfrac{\xi\eta}{b^2} \sum_{\mu} [K^\dagger K_\mu R_\mu R_Z (\Phi \Phi^\dagger) + \Phi^\dagger \Phi R_Z K_\mu K^\dagger - K_\mu R_\mu R_Z \Phi R_Z K^\dagger_\mu \Phi^\dagger - \Phi R_Z K_\mu R_\mu R_Z \Phi^\dagger K^\dagger_\mu] \\
& + \eta^2 [\Phi \Phi^\dagger - \mu^2]^2
\end{aligned}
\tag{2.13}
$$

where $\quad U_{\mu,\nu} = G_\mu R_\mu G_\nu R_\nu G^\dagger_\mu G^\dagger_\nu.$

Hence, we have got the lagrangian of gauge fields on each site, which is a function of coordinate (n, g). To obtain the physical action of gauge fields, we need to integrate $\mathcal{L}$ over spacetimes lattice and discrete group Z_2

$$
S_G = a^4 \sum_{n,g} \mathcal{L}.
$$

The exact expression of S_G will be given in next section. There are three types of gauge field in the generalized gauge theory, two of them—G_μ, K_μ should be vectorlike and Φ should be scalar. As a special case, it is possible to express K_μ in terms of G_μ and Φ, this will be discussed in construction of physical model in next section.

3 Generalized Gauge Theory on Lattice

We have constructed a generalized differential calculas on lattice and shown that the Higgs fields and the vectorlike fields may be introduced as generalized gauge fields on spacetime and discrete group as well as Yang-Mills fields on 4-dimmensional space time lattice. To deal with the model building in particle physics, fermions and their couplings to gauge fields must be included , we will first complete this issues in this section and then build the generalized gauge theory.

We now consider fermion fields on space time lattice and discrete group Z_2. In space time lattice theory, it is known that the fermion species doubling problem must be suppressed with some appropriate modification of the latticedized theory. One of the successful approaches as proposed by Wilson is so called Wilson fermion. In this paper, we first introduce Wilson-like fermion on space time lattice and discrete group Z_2 by taking into account the non-nearest-neighboure derivative of last section. Frist we introduce the free fermion lagrangian on lattice as follows

$$\begin{aligned} \mathcal{L}(n,g) = \; & \overline{\psi}(n,g)\gamma^\mu(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\psi(n,g) - \overline{\psi}(n,g)\partial_Z\psi(n,g) \\ & + \tfrac{g}{a}\overline{\psi}(n,g)\sum_\mu(\overrightarrow{\partial}_{Z\mu} + \overleftarrow{\partial}_{Z\mu})\psi(n,g), \end{aligned} \qquad g \in Z_2 \tag{3.1}$$

where $\psi(e) = \psi_L$, $\psi(Z) = \psi_R$ are left and right handed ferimons respectively. It is noted that the first and second terms in the lagrangian are similar as those of previous works[8, 9], if we want to include the the non-neighbouring differentiation $\partial_{Z\mu}$ in the lagrangian , the third term is in its most simple and non-trivial forms. The factor $\frac{1}{a}$ is need to cancall the doubling.

It is easy to show that we can get the action of Wilson fermion when integrating the lagrangian

(3.1),

$$\begin{aligned} S_F &= a^4 \sum_{n,g} \mathcal{L}(n,g) \\ &= a^3 \sum_{n,\mu} [\overline{\psi}(n)\gamma^\mu \psi(n+\mu) - \overline{\psi}(n+\mu)\gamma^\mu \psi(n)] + a^4 \sum_n \mu \overline{\psi}(n)\psi(n) \\ &\quad + \tfrac{q}{b} a^3 \sum_{n,\mu} [\overline{\psi}(n)\psi(n+\mu) + \overline{\psi}(n+\mu)\psi(n) - 2\overline{\psi}(n)\psi(n)]. \end{aligned} \tag{3.2}$$

So in the following, we will build lattice gauge theory using the lagrangian (3.1). The lagrangian (3.1) is invariant under the global transformations

$$\begin{array}{l} \psi(n) \to G\psi(n), \\ \overline{\psi}(n) \to \overline{\psi}(n) G^{-1} \end{array} \tag{3.3}$$

where G is an element of the gauge group. Similar to the reason that leads to the introduction of Yang-Mills fields, it is reasonable to require that the Lagrangian (3.1) be invariant under local gauge transformations $H(n,h),\ h \in Z_2$,

$$\psi(x,h) \to \psi(x,h)' = H(x,h)\psi(x,h). \tag{3.4}$$

Then the discrete partial dervative in (3.1) should be replace by covariant derivative as follows

$$\partial_Z \to D_Z, \quad \overrightarrow{\partial}_i \to \overrightarrow{D}_i, \quad \overleftarrow{\partial}_i \to \overleftarrow{D}_i, \quad i = \mu, Z\mu, \tag{3.5}$$

and $\overrightarrow{D}_i, \overleftarrow{D}_i$ and D_Z have the simple transformations under gauge transformation (3.3),

$$\begin{aligned} &\overrightarrow{D}_i\ \psi(n,g) \to [\overrightarrow{D}_i\ \psi(n,g)]' = H(n,g)\ \overrightarrow{D}_i\ \psi(n,g) \\ &\overline{\psi}(n,g)\ \overleftarrow{D}_i \to [\overline{\psi}(n,g)\ \overleftarrow{D}_i]' = \overline{\psi}(n,g)\ \overleftarrow{D}_i\ H^{-1}(n,g) \\ &D_Z\psi(n,g) \to [D_Z\psi(n,g)]' = H(n,g) D_Z \psi(n,g), \end{aligned} \tag{3.6}$$

where the covariant derivatives are formed as

$$\begin{aligned} \overrightarrow{D}_\mu\ \psi(n,g) &= (\partial_\mu + igA_\mu R_\mu)\psi(n,g) = \tfrac{1}{a}(U_\mu R_\mu - 1)\psi(n,g) \\ \overrightarrow{D}_{Z\mu}\ \psi(n,g) &= (\partial_{Z\mu} + igB_\mu R_\mu R_Z)\psi(n,g) = \tfrac{1}{b}(K_\mu R_\mu R_Z - 1)\psi(n,g) \\ D_Z\psi(n,g) &= (\partial_Z + \lambda\phi R_Z)\psi(n,g) = \lambda(\Phi R_Z - \tfrac{\mu}{\lambda})\psi(n,g), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} G_\mu &= 1 + igaA_\mu, \mu = 1,2,3,4 \\ \Phi &= \tfrac{\mu}{\lambda} + \phi \\ K_\mu &= 1 + bB_\mu, \mu = 1,2,3,4 \end{aligned}$$

correspondingly, we have the expression of $\overleftarrow{D}_i$ as

$$\begin{aligned}\overline{\psi}(n,g)\overleftarrow{D}_\mu &= \tfrac{1}{a}\overline{\psi}(n,g)(\overleftarrow{R}_\mu U^\dagger_\mu - 1)\\ \overline{\psi}(n,g)\overleftarrow{D}_{Z\mu} &= \tfrac{1}{b}\overline{\psi}(n,g)(\overleftarrow{R}_{Z\mu} K^{-1}_\mu - 1).\end{aligned}$$

Then the transformation law (3.3) are satisfied if the generalized gauge fields U_μ, K_μ and Φ have the following properties:

$$\begin{aligned}G_\mu &\to HG_\mu(R_\mu H^{-1})\\ \Phi &\to H\Phi R_Z H^{-1}\\ K_\mu &\to HK_\mu(R_\mu R_Z H^{-1}).\end{aligned} \tag{3.8}$$

Similar to the usual gauge theory, the covariant dreivative is equivalent to the covariant exterior derivative $D = d + A$ and

$$Df = D_\mu f dx^\mu + D_{Z\mu}\chi^\mu + D_Z f\chi. \tag{3.9}$$

Thus from (3.1), the gereralized gauge invariant lagrangian for fermions should be,

$$\begin{aligned}\mathcal{L}_F(n,g) = \;& \overline{\psi}(n,g)\gamma^\mu(\overrightarrow{D}_\mu - \overleftarrow{D}_\mu)\psi(n,g) - \overline{\psi}(n,g)D_Z\psi(n,g)\\ &+\tfrac{g}{a}\overline{\psi}(n,g)\sum_\mu(\overrightarrow{D}_{Z\mu} + \overleftarrow{D}_{Z\mu})\psi(n,g),\end{aligned} \tag{3.10}$$

and in terms of the expessions of D_i, D_Z, it may be written as:

$$\begin{aligned}\mathcal{L}(n,g) = \;& \tfrac{1}{a}\sum_\mu[\overline{\psi}(n,g)\gamma^\mu G_\mu(n,g)\psi(n+\mu,g) - \overline{\psi}(n+\mu,g)\gamma^\mu G^\dagger_\mu\psi(n,g)]\\ &+\lambda\overline{\psi}(n,g)\Phi(n,g)\psi(n,gZ)\\ &+\tfrac{g}{ab}\sum_\mu[\overline{\psi}(n,g)K_\mu\psi(n+\mu,gZ) + \overline{\psi}(n+\mu,gZ)K^\dagger_\mu\psi(n,gZ)].\end{aligned} \tag{3.11}$$

In the following calculaiton, we set

$$\begin{aligned}G_\mu(n,e) = G^L_\mu, &\quad G_\mu(n,Z) = G^R_\mu(n)\\ \Phi(n,e) = \Phi(n), &\quad \Phi(n,Z) = \Phi(n)\\ K_\mu(n,e) = K^1_\mu(n), &\quad K_\mu(n,Z) = K^2_\mu(n).\end{aligned} \tag{3.12}$$

After integrating lagrangian (3.11) over spacetime and discrete group Z_2, we get the covariant action of fermoin sector,

$$
\begin{aligned}
S_F = \;& a^3 \sum_{n,\mu} [\overline{\psi}_L(n)\gamma^\mu G^L_\mu(n)\psi_L(n+\mu,g) - \overline{\psi}_L(n+\mu,g)\gamma^\mu G^{L\dagger}_\mu \psi_L(n,g) \\
&\quad + \overline{\psi}_R(n)\gamma^\mu G^R_\mu(n)\psi_R(n+\mu,g) - \overline{\psi}_R(n+\mu,g)\gamma^\mu G^\dagger R_\mu \psi_R(n,g)] \\
&+\tfrac{g}{b} a^3 \sum_{n,\mu} [\overline{\psi}_L(n) K^1_\mu \psi_R(n+\mu) + \overline{\psi}_R(n) K^2_\mu \psi_L(n+\mu) + h.c.]] \\
&+\lambda a^4 \sum_n [\overline{\psi}_L(n)\Phi(n)\psi_R(n) + \overline{\psi}_R(n)\Phi^\dagger(n)\psi_L(n)].
\end{aligned}
\tag{3.13}
$$

Using the results of geralized differential calculus on space time lattice and discrete group Z_2 of last section, we may introduce the action of gauge fields from the curvature as follows:

$$
\begin{aligned}
S_G = \;& \sum_{n,\mu,\nu} \frac{1}{2}[Tr(1-U^L_{\mu,\nu}) + Tr(1-U^R_{\mu,\nu})] \\
&+\tfrac{\xi^2}{b^4} a^4 \sum_{n,\mu,\nu} \frac{1}{2} Tr[K^{1\dagger}_\mu K^1_\mu R_\mu (K^2_\nu K^{2\dagger}_\nu) - K^{1\dagger}_\mu K^1_\nu R_\nu K^2_\mu R_\mu K^{2\dagger}_\nu] \\
&+\tfrac{\xi}{a^2 b^2} a^4 \sum_{n,\mu,\nu} Tr[2(K^1_\mu K^{1\dagger}_\mu + K^2_\mu K^{2\dagger}_\mu) - (G^L_\mu R_\mu K^1_\nu R_\nu G^{R\dagger}_\mu K^{1\dagger}_\mu + G^R_\mu R_\mu K^2_\nu R_\nu G^{L\dagger}_\mu K^{2\dagger}_\mu + h.c.)] \\
&+\tfrac{\xi\eta}{a^2} a^4 \sum_{n,\mu} [4\Phi\Phi^\dagger - 2(G^L_\mu R_\mu \Phi G^{R\dagger}\Phi^\dagger + h.c.)] \\
&+\tfrac{\xi\eta}{b^2} a^4 \sum_{n,\mu} [K^{1\dagger}_\mu K^1_\mu R_\mu(\Phi^\dagger\Phi) + \Phi^\dagger\Phi K^2_\mu K^{2\dagger}_\mu + K^{2\dagger}_\mu K^2_\mu R_\mu(\Phi\Phi^\dagger) + \Phi\Phi^\dagger K^1_\mu K^{1\dagger}_\mu \\
&\qquad - 2K^1_\mu R_\mu \Phi^\dagger K^{2\dagger}_\mu \Phi^\dagger - \Phi K^2_\mu R_\mu \Phi K^{1\dagger}_\mu] \\
&+\eta^2 a^4 \sum_n 2Tr(\Phi\Phi^\dagger - \frac{\mu^2}{\lambda^2})^2.
\end{aligned}
\tag{3.14}
$$

So far, we have completed the constructin of generalized gauge theory on spacetime lattice and discrete group Z_2. In next section, we will show the Smit-Swift model on lattice is a special case of the model we dicussed above.

4 Smit-Swift Model

We have finished the construction of physical model in last subsection, three kinds of gauge field—G_μ, K_μ, Φ are introduce in the approach, but, in realistic lattice model, there are only two kinds of gauge fields —Yang-Mills fields G_μ and Higgs fields Φ, so it is desired to express the third gauge fields K_μ in terms of G_μ and Φ or we treat the third connection as a combination of the others.

By the requirement of gauge transformation properties (3.8), the most general experssion of H_μ in terms of G_μ and Φ is the linear combination of two terms $G_\mu R_\mu \Phi$ and $\Phi R_Z G_\mu$, and the most general expression is

$$\begin{aligned} K_\mu^1 &= \alpha_1 \Phi G_\mu^R + \beta_1 G_\mu^L R_\mu \Phi \\ K_\mu^2 &= \alpha_2 \Phi^\dagger G_\mu^L + \beta_2 G_\mu^R R_\mu \Phi^\dagger. \end{aligned} \tag{4.1}$$

In the following, we will find that Smit-Swift model may be reconstructed when a special case of (4.1) is taken into account,

$$K_\mu^1 = f\Phi G_\mu^R, \quad K_\mu^2 = f G_\mu^R R_\mu \Phi^\dagger, \tag{4.2}$$

where f is a free parameter.

Substitute the expression of K_μ^1 and K_μ^2 into the fermion action (3.13), one obtains

$$\begin{aligned} S_F = \; & a^3 \sum_{n,\mu} [\overline{\psi}_L(n)\gamma^\mu G_\mu^L(n)\psi_L(n+\mu,g) - \overline{\psi}_L(n+\mu,g)\gamma^\mu G_\mu^{L\dagger}\psi_L(n,g) \\ & \qquad + \overline{\psi}_R(n)\gamma^\mu G_\mu^R(n)\psi_R(n+\mu,g) - \overline{\psi}_R(n+\mu,g)\gamma^\mu G^\dagger R_\mu \psi_R(n,g)] \\ & + r a^3 \sum_{n,\mu} [\overline{\psi}_L(n)\Phi G_\mu^R \psi_R(n+\mu) + \overline{\psi}_R(n) G_\mu^R \Phi(n+\mu)\psi_L(n+\mu) + h.c.]] \\ & + \lambda a^4 \sum_n [\overline{\psi}_L(n)\Phi(n)\psi_R(n) + \overline{\psi}_R(n)\Phi^\dagger(n)\psi_L(n)], \end{aligned} \tag{4.3}$$

where $r = \frac{g}{b} f$.

It is easy to show that the Higgs potential reach its minimu when $|\Phi| = \frac{\mu}{\lambda}$. To simplify the problem(without throwing away any important physics [12, 14]), we freeze out the radial model of the Higgs fields, working with fields with fixed norm $\Phi\Phi^\dagger = \frac{\mu^2}{\lambda^2}$. Thus we shall be dealing with fields that are strictly compact. Under above assumption, we obtain the Lagrangian of gauge field by throwing away constant numbers as following:

$$\begin{aligned} S_G = \; & \tfrac{1}{2} Tr \sum_{n,\mu,\nu} U_{\mu,\nu}^L + \frac{1}{2}(1 + \xi^2 f^4 \frac{\mu^4}{\lambda^4}\frac{a^4}{b^4}) Tr \sum_{n,\mu,\nu} U_{\mu,\nu}^R \\ & + \tfrac{f^2}{b^2} a^2 \xi \sum_{n,\mu,\nu} Tr[U_{\mu,\nu} S_\mu + \tilde{U}_{\mu,\nu}^R R_\nu S_\mu^\dagger) + h.c.]] \\ & + 2\xi\eta a^2 \sum_{n,\mu} (S_\mu + S_\mu^\dagger)], \end{aligned} \tag{4.4}$$

where $S_\mu = \Phi^\dagger G_\mu(e) R_\mu \Phi G_\mu^\dagger(Z)$, $\tilde{U}^R_{\mu,\nu} = G_\nu^{R\dagger} G_\mu R_\mu G_\nu R_\nu G_\mu^{R\dagger}$. It is easy to show that $\tilde{U}^R_{\mu,\nu} = U^R_{\mu,\nu}$ if the gauge fields U^R is Abelian. Parameters f, ξ, η are free, so up to a proportionate constant, we may write the lagrangian in the following form:

$$\begin{aligned} S_G = \; & \beta_1 Tr \sum_{n,\mu,\nu} U^L_{\mu,\nu} + \beta_2 Tr \sum_{n,\mu,\nu} U^R_{\mu,\nu} \\ & + \beta_3 a^2 \sum_{n,\mu,\nu} Tr[U_{\mu,\nu} S_\mu + \tilde{U}^R_{\mu,\nu} R_\nu S_\mu^\dagger + h.c.] \\ & + \beta_H a^2 \sum_{n,\mu} (S_\mu + S_\mu^\dagger)], \end{aligned} \tag{4.5}$$

where $\beta_1, \beta_3, \beta_H$ are free parameters and $\beta_2 = \frac{\beta_1^2 + \beta_3^2 a^4}{\beta_1}$.

There is a high order terms appear in the lagrangian, which is different from ordinary lattice gauge theory, whether this term can lead to new physics need to be studied future.

4.1 A Toy Model

Now we consider a "toy model" in which left-handed fields transform according to the fundamental representation of a gauge group—say $SU(2)$—and there is a right-handed partner which transforms trivially. Hence, the gauge fields G_μ and Φ on each sector should be:

$$\begin{aligned} & G^L_\mu = U_\mu, \quad G^R_\mu = 1 \\ & K^1_\mu = f\Phi, \quad K^2_\mu = f R_\mu \Phi. \end{aligned} \tag{4.6}$$

As a direct results of (4.3)

$$\begin{aligned} S_F = \; & a^4 \sum_{n,\mu} [\overline{\psi}_L(n) \gamma^\mu U_\mu(n) \psi_L(n+\mu, g) - \overline{\psi}_L(n+\mu, g) \gamma^\mu U_\mu^\dagger \psi_L(n, g) \\ & \qquad + \overline{\psi}_R(n) \gamma^\mu \psi_R(n+\mu, g) - \overline{\psi}_R(n+\mu, g) \gamma^\mu \psi_R(n, g)] \\ & + \lambda a^4 \sum_n [\overline{\psi}_L(n) \Phi(n) \psi_R(n) + \overline{\psi}_R(n) \Phi^\dagger(n) \psi_L(n)] \\ & + r a^3 \sum_{n,\mu} [\overline{\psi}_L(n) \Phi \psi_R(n+\mu) + \overline{\psi}_R(n) \Phi^\dagger(n+\mu) \psi_L(n+\mu) + h.c.] \end{aligned} \tag{4.7}$$

Now freezing out the radial model of the Higgs fields and throwing away some constant, we get a simple mode

$$S_G = \beta \sum_{n,\mu,\nu} [Tr U_{\mu,\nu})] + \beta_H Tr[\Phi^\dagger U_\mu R_\mu \Phi + h.c.]. \tag{4.8}$$

where β and β_H are free parameters. This toy model is study in many works, such as [14].

4.2 $SU(2)\times U(1)$ Electroweak Model on Lattice

With all previous points in mind, a construction for the $SU(2)\times U(1)$ electroweak theory on the lattice is now available. Consider just the leptonic first generation sector: electron and electron neutrino and define the fields as follows:

$$
\begin{aligned}
&\psi(e) = L = \begin{pmatrix} v_L \\ e_L \end{pmatrix}, && \psi(Z) = \begin{pmatrix} R_2 \\ R_1 \end{pmatrix} = \begin{pmatrix} v_R \\ e_R \end{pmatrix} \\
&G_\mu(e) = G_\mu^L = U_\mu V_\mu, && G_\mu(Z) = G_\mu^R = \begin{pmatrix} 1 & \\ & V'_\mu \end{pmatrix} \\
&\Phi(e) = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_+^* & \phi_0 \end{pmatrix}, && \Phi(Z) = \begin{pmatrix} \phi_0 & -\phi_+ \\ \phi_+^* & \phi_0^* \end{pmatrix} \\
&K_\mu^1 = \Phi G_\mu(Z) f, && K_\mu^2 = f G_\mu(Z) R_\mu \Phi^\dagger
\end{aligned}
\tag{4.9}
$$

where $U_\mu(n) = exp[iga\tau\cdot A_\mu(n)]$ is $SU(2)$ gauge field and $V_\mu(n) = exp[-\frac{1}{2}ig'aB_\mu(n)]$, $V'_\mu(n) = exp[-ig'aB_\mu(n)]$ are $U(1)$ gauge field, $f = \begin{pmatrix} f_2 & \\ & f_1 \end{pmatrix}$ is introduced as non-trivial coupling constant which is consistent with gauge transformation properties (3.8).

Using the results of last section and those assigments of (4.9), we have the action for the fermion part,

$$
\begin{aligned}
L = \; & a^3 \sum_n [\overline{L}(n)\gamma^\mu U_\mu V_\mu L(n+\mu) - \overline{L}(n+\mu)\gamma^\mu U_\mu^\dagger V_\mu^\dagger L(n) \\
& \quad + \overline{R}_1(n)\gamma^\mu V'_\mu R_1(n+\mu) - \overline{R}_1(n+\mu)\gamma^\mu V_\mu'^\dagger R_1(n) \\
& \quad + \overline{R}_2(n)\gamma^\mu R_1(n+\mu) - \overline{R}_1(n+\mu)\gamma^\mu R_1(n)] \\
& + a^4 \sum_n \lambda[\overline{L}(n)\phi R_1 + \overline{R}_1(n)\phi^\dagger L(n) + \overline{L}(n)\,\tilde{\phi}\, R_2 + \overline{R}_2(n)\,\tilde{\phi}^\dagger\, L(n)] \\
& + a^3 \sum_{n,\mu} r_1[\overline{L}(n)V'_\mu(n)\phi R_1(n+\mu) + \overline{R}_1(n)V'_\mu(n)\phi^\dagger(n+\mu)L(n+\mu) + h.c.] \\
& a^3 \sum_{n,\mu} r_2[\overline{L}(n)\,\tilde{\phi}\, R_2(n+\mu) + \overline{R}_2(n)\,\tilde{\phi}^\dagger\,(n+\mu)L(n+\mu) + h.c.]
\end{aligned}
\tag{4.10}
$$

where $\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}$ and $\tilde{\phi} = \begin{pmatrix} \phi_0^* \\ -\phi_+^* \end{pmatrix}$, two constants are $r_1 = f_1\frac{g}{b}$, $r_2 = f_2\frac{g}{b}$.

Substitute the assignments of fields (4.9) into action of bosonic sector(4.5), we have

$$\begin{aligned} S_G = &\ \beta_1 Tr \sum_{n,\mu,\nu} U_{\mu,\nu} + \beta_2 Tr \sum_{n,\mu,\nu} V'_{\mu,\nu} + \beta_H \sum_{n,\mu} Tr S_\mu \\ & + \beta_3 Tr \sum_{n,\mu,\nu} \left[\begin{pmatrix} (\frac{r_2}{r_1})^2 & \\ & V'_{\mu,\nu} \end{pmatrix} (S_\mu + R_\nu S_\mu^\dagger) + \begin{pmatrix} (\frac{r_2}{r_1})^2 & \\ & V'^{\dagger}_{\mu,\nu} \end{pmatrix} (S_\mu^\dagger + R_\nu S_\mu) \right], \end{aligned} \tag{4.11}$$

where

$$S_\mu = \Phi^\dagger U_\mu R_\mu \Phi \begin{pmatrix} V_\mu^\dagger & \\ & V_\mu \end{pmatrix}.$$

It is easy to show that

$$Tr[S_\mu] = \phi^\dagger(n) U_\mu(n) V_\mu^\dagger \phi(n+\mu) + \phi^\dagger(n+\mu) U_\mu^\dagger V_\mu^\dagger \phi(n), \tag{4.12}$$

which is the kinetic term of Higgs field on lattice in Smit-Swift model[14].

At last we can write the lagrangian of bosonic part as following:

$$\begin{aligned} S_G = &\ \beta_1 Tr \sum_{n,\mu,\nu} U_{\mu,\nu} V_{\mu,\nu} + \beta_2 Tr \sum_{n,\mu,\nu} V'_{\mu,\nu} \\ & + \beta_H a^2 \sum_{n,\mu} [\phi^\dagger(n) U_\mu(n) V_\mu^\dagger(n) \phi(n+\mu) + \phi^\dagger(n+\mu) U_\mu^\dagger(n) V_\mu(n) \phi(n)] \\ & + \beta_3 a^2 Tr \sum_{n,\mu,\nu} \left[\phi^\dagger(n) U_\mu(n) V_\mu^\dagger(n) \Phi(n+\mu) [V'_{\mu,\nu}(n) + V'^{\dagger}_{\mu,\nu}(n-\nu)] + h.c. \right], \end{aligned} \tag{4.13}$$

Therefore, we have constructed a chiral model on lattice with the knowledge of noncommutative differential geometry. In the approximation of small parameter a, $Tr(U_{\mu,\nu} V_{\mu,\nu}) \sim Tr U_{\mu,\nu} + 2V_{\mu,\nu}$, so most of the results are the same with Smit-Swift model after we freeze the radial of higgs field except one term more in the bosonic sector. The naive continuum limit of this term is vanish. However, it is needed to be studied furture whether this term may lead to any result in physics. In recent years study, it is found that previous chiral models on lattice always meet some problems [15], so it is desired to develop new theory to overcome those difficulties. We have given a general rule to construct the chiral model on lattice. This may help us to develop the previous models and build new models. In this approach, it is also possible to study the chiral models on lattice in other schems, such as staggered fermion and domail-wall ect. These topics will be studied in our comming papers.

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References

[1] A. Dimakis, F. Müller-Hoissen and T. Striker .F. J. Phys.A: Math. Gen. **26** 1927 (1993).

[2] A. Dimakis and F. Müller-Hoissen .F. J. Phys.A: Math. Gen. **27** 3159 (1994).

[3] A. Connes and J. Lott, Nucl. Phys. (Proc. Suppl.) **B18**, 44 (1990).

[4] A. Connes, Noncommutative Geometry, Academic Press 1994.

[5] R. Coquereaux, G. Esposito-Farése and G Vaillant, Nucl Phys **B353** 689 (1991).

[6] A. H. Chamseddine, G Felder and J. Fröhlich, Phys. Lett. **296B** (1993) 109; Nucl. Phys. **B395** 672; Commu.Math.Phys. **155** (1993) 205.

[7] A. Sitarz, J. Geom. Phys.15(1995)123; Phys. Lett. **308B**(1993) 311.

[8] Haogang Ding, Hanying Guo, Jianming Li and Ke Wu, Commun. Theor.Phys. **21** 85(1994).

[9] Haogang Ding, Hanying Guo, Jianming Li and Ke Wu, Z. Phys.**C64** 512(1994) ;J. Phys. A:Math. Gen. **27** L231(1994).

[10] E.Álvarez, J.M. Gracia-Bondia and C.P. Martin, Phys. Lett. **B306** 53 (1993).

[11] K.G. Wilson , Phys. Rev. **D10**, 2445(1974); New Phenomena in Subnuclear Physics, **Part A**, 69 (1975), Edited by A. Zichichi, Plenum Press new York and London.

[12] E.Pradkin and S.H. Schenker, Phys. Rev. **D19** 3682(1979).

[13] J. Smit, Acta Physica Polonica **B17**,531(1986).

[14] P.D.V. Swift, Phys. Lett. **B378**, 652(1992).

[15] D.N. Petcher, Nucl. Phys. **B30**(Proc. Suppl.) 50(1993).